Empirical Functions of Two Variables, and Dicas's Hydrometer Correction

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Introduction

A general function $z(x, y)$ can be implemented on a slide rule if there exist scale functions $f(x)$, $g(y)$, and $h(z)$ such that

$$h\big(z(x, y)\big) = f(x) + g(y). \qquad (1)$$

Here h is a function that maps z to a distance along the slide rule, f maps x to distance, and g maps y to distance. Of course, the mapping (1) is not unique: one may introduce additive constants a and b, and a nonzero multiplicative constant c, to give a different but equivalent slide rule mapping:

$$\tilde{h}\big(z(x, y)\big) = \tilde{f}(x) + \tilde{g}(y)$$

with

$$\tilde{f}(x) = c\, f(x) + a,$$
$$\tilde{g}(y) = c\, g(y) + b,$$

and

$$\tilde{h}(z) = c\, h(z) + a + b.$$

The standard logarithmic slide rule operates on this approach for multiplication ($z = xy, f = g = h = \ln$) and division ($z = x/y, f = -g = h = \ln$). Empirical formulas with non-logarithmic scales obey Eq. (1) also.

The first empirical function of two variables put on a slide rule is believed to be John Dicas's hydrometer correction (Fig. 1). Understanding this particular slide rule is what motivated the general analysis presented below.

Other authors have been concerned with general two-variable functions from different perspectives. Alfeld[1], Szalkai[2,3,4] and Hoffman[4] investigate the scales associated with two-variable functions with Eq. (1) in mind, but their focus is on expressions $z(x, y)$ that can be written in closed form. That approach leads to a search for associated sets of four functions (z, f, g, h), all expressible as exact mathematical functions.

In contrast, the approach taken here is to explore the mathematical implications of Eq. (1) without regard to specific instances. This approach leads to constraints on the function $z(x, y)$ by itself, without any knowledge or consideration of the mapping functions f, g, or h. If z satisfies the given conditions, the mapping functions are easily determined to within additive and multiplicative constants. The benefit of this approach is that it is truly general and can be applied to tabulated or algorithmically-defined functions. The difficulty with the approach is that its explanation relies on a bit of calculus and its evaluation benefits from numerical methods, particularly for non-analytical functions.

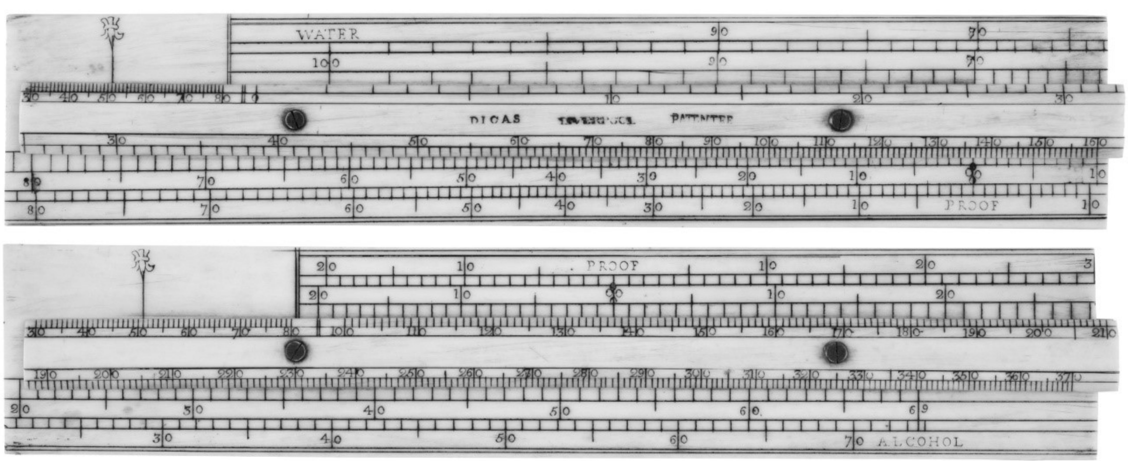

FIGURE 1. A slide rule for the temperature correction of hydrometer readings by John Dicas, ca 1780. Temperature in °F is placed opposite the *fleur de lis* (top left). The uncorrected hydrometer reading from 0 to 370 is located on the slider, and the corrected alcohol strength is read on the frame scales. The inner scale is believed to be Dicas's and the outer scale Clarke's.

The necessary properties of $z(x, y)$

The key property of slide rule mapping Eq. (1) is its "additive separability." On the left-hand side, the independent variables x and y are inseparably linked by the abstract function z. But the right-hand side consists of a function of x only added to a function of y only. Differentiation of Eq. (1) by x will eliminate the y-dependence on the right-hand side of Eq. (1), and differentiation by y eliminates the x-dependence. The mixed second derivative eliminates the right-hand side altogether:

$$\frac{\partial^2}{\partial x \partial y} h\big(z(x, y)\big) = 0,$$

or,

$$h'' z_x z_y + h' z_{xy} = 0$$

after expanding the left-hand side. Rearranging this result separates the h-dependence to one side:

$$\frac{d}{dz} \ln h'(z) = -\frac{z_{xy}}{z_x z_y}.$$

The left-hand side of this equation depends on z only, therefore the right-hand side must also depend only on z. The partial derivative of the right-hand side with respect to x, while holding z constant, must be zero. With some manipulation this statement gives the result

$$z_y \frac{\partial}{\partial x}\bigg|_z \frac{z_{xy}}{z_x z_y} = \frac{\partial^2}{\partial x \partial y}\left[\ln z_x - \ln z_y\right] = 0.$$

Recalling that the second mixed derivative of an additively separable function is zero, we conclude that the term in square brackets is additively separable. Rearranging somewhat,

$$\ln\left(-\frac{z_x}{z_y}\right) = \ln\left(\frac{\partial y}{\partial x}\bigg|_z\right) = u(x) + v(y)$$

for some functions $u(x)$ and $v(y)$. Alternatively,

$$\frac{\partial y}{\partial x}\bigg|_z = e^{u(x)} e^{v(y)}. \tag{2}$$

This is the first key result: knowing nothing about f, g, or h, we can say that a function $z(x, y)$ can somehow be implemented on a slide rule only if contours of constant z have slopes obeying the multiplicative decomposition indicated by Eq. (2). If the functions $u(x)$ and $v(y)$ are continuous, then the slopes cannot change sign. (The slopes can be negative using Euler's identity, $\exp(\pi i) = -1$, to introduce a constant imaginary part to the sum $u + v$, but the slope cannot change between positive and negative if $u + v$ is continuous.) It will be seen that the practical requirement that mappings be monotonic also forbids changes of sign in Eq. (2).

This is already a powerful result in that it indicates the existence of an unbounded number of slide-rule-capable functions. It also provides a simple test of suitability. For example, if Eq. (2) holds true, then the ratio of slopes taken at different x values,

$$\frac{\frac{\partial y}{\partial x}\Big|_z (x_1, y)}{\frac{\partial y}{\partial x}\Big|_z (x_2, y)} = e^{u(x_1) - u(x_2)}, \tag{3}$$

is independent of y. Likewise, a ratio of slopes evaluated at different y values must be independent of x.

The mapping functions f, g, and h

Eq. (2) describes the slopes of the z function but it does not relate functions u or v to the value of the function z. However, if point (x, y) and point (x', y') lie on the same contour of constant z, then integrating Eq. (2) gives

$$\int_x^{x'} e^{u(x)} dx = \int_y^{y'} e^{-v(y)}\, dy. \tag{4}$$

If, furthermore, the points (x', y') lie on some fiducial curve where z is known then Eq. (4) would be a way to determine the value of z at any point (x, y). For instance, suppose the primed points were on a fiducial line $y' = m\, x' + b$ on which $x' = \omega(z)$, then
Eq. (4) could be rearranged to give

$$\underbrace{\int_{y_0}^{m\omega(z)+b} e^{-v(y)} dy - \int_{x_0}^{\omega(z)} e^{u(x)} dx}_{h(z)} = \underbrace{\int_x^{x_0} e^{u(x)} dx}_{f(x)} + \underbrace{\int_{y_0}^{y} e^{-v(y)} dy}_{g(y)}, \tag{5}$$

which has the form of the additively separable Eq. (1). In this equation the constants x_0 and y_0 are arbitrary.

This is the second main result: exponential integrals of the functions $u(x)$ and $-v(y)$ determine the mapping functions f, g, and h. The particulars of the fiducial curve depend somewhat on the function being modeled. Eq. (5) also shows that continuity in u and v assures monotonicity of the scale functions f and g.

Multiplication

If this analysis is correct then it should deduce that $f = g = h = \ln$ for $z(x,y) = xy$. For this multiplication function,

$$\left.\frac{\partial y}{\partial x}\right|_z = -\frac{y}{x} = e^{\pi i + \ln y - \ln x}$$

using Euler's identity to change the sign, so one could choose $u(x) = \pi i - \ln x$ and $v(y) = \ln y$. Let us also choose $y' = 1$ as the fiducial line, on which $x' = z$: in Eq. (5) let $m = 0$, $b = 1$, and $\omega(z) = z$. Then,

$$h(z) = \int_{y_0}^{1} \frac{dy}{y} + \int_{x_0}^{z} \frac{dx}{x} = \ln z - \ln x_0 - \ln y_0$$

$$f(x) = -\int_{x}^{x_0} \frac{dx}{x} = \ln x - \ln x_0$$

$$g(y) = \int_{y_0}^{y} \frac{dy}{y} = \ln y - \ln y_0.$$

We obtain the expected slide rule mapping $\ln(xy) = \ln(x) + \ln(y)$ upon simplification or by choosing $x_0 = y_0 = 1$.

Had one chosen instead $u(x) = -\ln x$ and $v(y) = \pi i + \ln y$ then one would have obtained the essentially identical result $-\ln(xy) = -\ln(x) - \ln(y)$. The diagonal fiducial $y = \sqrt{z}$ with $x = \sqrt{z}$ likewise gives the expected answer.

Empirical functions: Dicas's hydrometer correction

In the multiplication example the function z being modeled had a simple analytical expression which yielded analytical expressions for the key functions $u(x)$ and $v(y)$. Sometimes z is known only from a table of numbers, or a complex numerical algorithm. In such cases a purely mathematical analysis fails and one must resort to numerical approximation.

To illustrate the approach, I will focus on the specific example of the temperature correction of a hydrometer reading. Hydrometers operate on Archimedes' principle of buoyancy. A hydrometer will sit high in a dense fluid, but sink low in a light one. To the extent that alcohol affects the density of a water-alcohol solution, a hydrometer's reading is an indication of alcoholic strength. Hydrometers have been used since Boyle in the 17th Century to assess alcoholic spirits.[5] Temperature also affects a hydrometer's reading. It can cause expansion or contraction of the fluid, and a different expansion or contraction of the hydrometer body, and both of these changes modify the instrument's buoyancy. Therefore, to make an accurate determination of alcoholic strength one must account for the effect of temperature. Clarke's hydrometer approximated this temperature correction in the 18th Century by application of discrete so-called weather weights.[6]

In 1780 John Dicas, a Liverpool brandy merchant, patented an improved alcohol hydrometer that placed the water-alcohol continuum on a continuous scale from 0 to 370, and which implemented a continuous temperature correction using a slide rule.[7] Dicas's hydrometer and its slide rule were adopted by the U.S. Treasury at the recommendation of Alexander Hamilton in 1790,[8] and it remained the U.S. standard until about 1851.[9]

Dicas's instruments thus played a key role in the early days of the American republic. His invention also played a pivotal role in the history of slide rules. His temperature-correcting slide rule is (to my knowledge) the first implementation of an empirical function of two variables.[10] Until his 1780 patent, slide rules were restricted to applications of logarithmic scales and one-dimensional functions which are essentially just look-up tables. The "segment lying" and "segment standing" scales of Everard type gaugers' slide rules are examples of these one-dimensional functions.[11] These scales can be determined using calculus,[12] but the lines of numbers were most likely empirically derived, at least on early slide rules.[13]

The idea behind Dicas's application became widely adopted. In 1803, a competition was held to replace Clarke's hydrometer as the UK standard. Entries by Mary Dicas (John's daughter), John and George Quin, Robert Atkins, and by John Dring and William Fage all included temperature-correcting slide rules. The winner of the competition, Bartholomew Sikes, was the only entrant to rely on a book of tables instead of a slide rule.[14,15] However, during the 19th Century makers of Sikes' hydrometer including Bates, Buss, Farrow and Jackson, Gill, and Long, all produced slide rules for Sikes's hydrometers. Dicas adapted the idea to the lactometer and saccharometer, and Thomas Thomson's Allan saccharometer used a slide rule for temperature correction. In the 20th Century, Francis

Charles Farmar invented a slide rule for spirits dealers that featured an empirical dilution calculation.

Nothing is known about the empirical data John Dicas relied on, or how he used it to construct his slide rules. We can gain an appreciation for his accomplishment, however, using modern data and computational algorithms to interpret Eq. (5). For data, an algebraic expression exists for the density of water-alcohol mixtures as a function of temperature and the mass fraction of ethanol (OIML).[16] Using 51×10^{-6}/K as the thermal expansion of copper accounts for the instrument effect. Let x be the measured or apparent alcoholic proof at measurement temperature y. Let $z(x, y)$ be the true alcoholic proof, or the value that would have been measured had the temperature been the reference temperature of 55°F.[17] The natural choice of fiducial line is $y' = 55$°F where $z = x'$ (i.e., $m = 0$, $b = 55$°F, $\omega = z$). The calculation $z(x, y)$ is outlined elsewhere.[12] (For Sikes's hydrometers the reference temperature is 51°F.)

Given this starting point, the challenge is the determination of $u(x)$ and $v(y)$. This is done computationally by expanding the function in a basis with compact support, such as b-splines:

$$u(x) = \sum_m c_m \; p_m(x)$$
$$v(y) = \sum_n d_n q_n(y) \quad .$$

In these expansions the basis functions p and q are fixed step-like or triangle-shaped functions centered at different points such that a suitably scaled combination of them can approximate any function. Let $\tilde{z}(x, y)$ be the hydrometer correction computed from $u(x)$ and $v(y)$ using Eq. (5). The true function $z(x, y)$ might not satisfy Eq. (2), and might not, therefore, be exactly implemented on a slide rule. The approximation $\tilde{z}\,(x, y)$, being determined by Eq. (5), may be "exactly" implemented on a slide rule, but it might not model the behavior of the hydrometer exactly. The sets of coefficients $\{c\}$ and $\{d\}$ are chosen to make $\tilde{z}$ be as close as possible to z in a least squares sense. Computationally this is done by iteratively solving the numerical Taylor series approximation

$$z(x_i, y_j) = \tilde{z}(x_i, y_j) + \sum_m \frac{\partial \tilde{z}(x_i, y_j)}{\partial c_m} \Delta c_m + \sum_n \frac{\partial \tilde{z}(x_i, y_j)}{\partial d_n} \Delta d_n$$

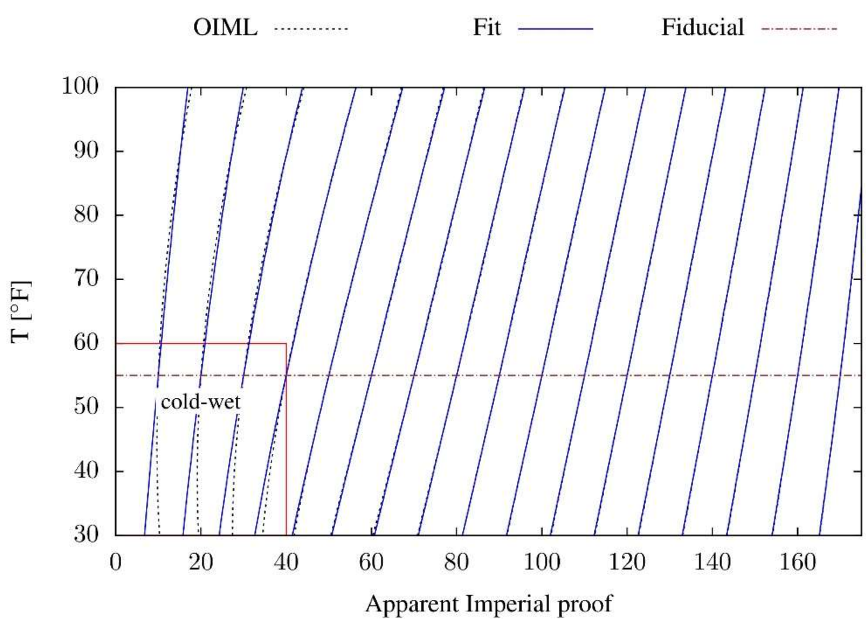

FIGURE 2. The black dashed curves represent contours of constant true proof calculated from the OIML data set. The slopes of these curves are generally positive, but in the cold-wet region the slopes can be negative. Solid blue curves are contours of a least squares best fit to the slide rule compatible model (Eq. (5)). The brown dot-dash line is the 55°F fiducial curve.

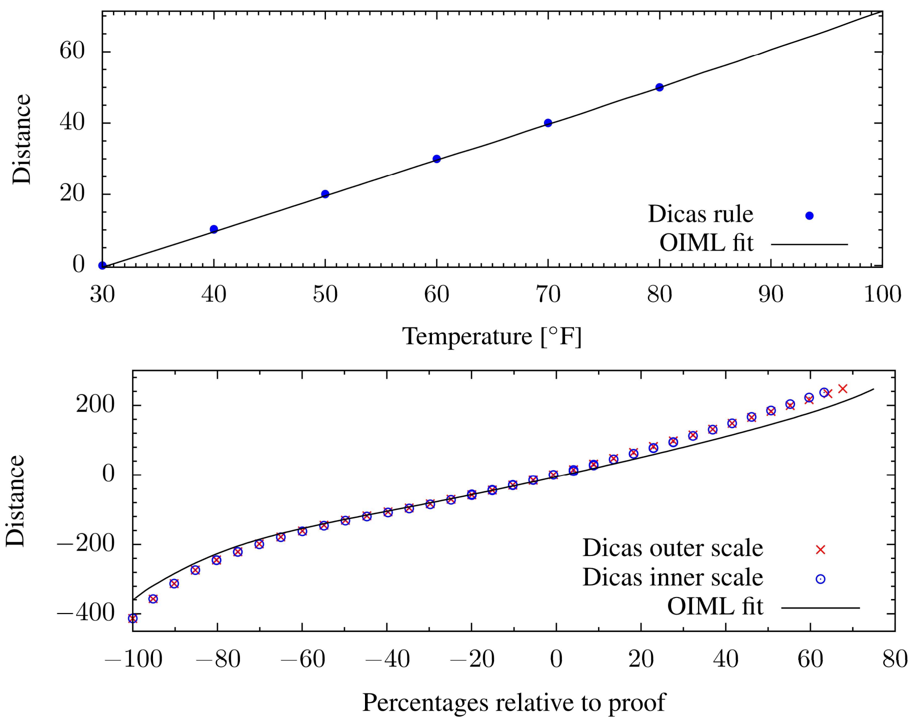

FIGURE 3. The solid curves are the mapping functions for temperature (top, g) and proof (bottom, f, h) computed during the fitting process. The superimposed points come from the slide rule in Fig. 1. The model scale is chosen so that the 30 to 80°F distances match exactly.

for sets of corrections $\{\Delta c\}$ and $\{\Delta d\}$ to the weighting coefficients. When the iteration converges, $\tilde{z}$ is a least squares best fit to z on the chosen quadrature points (indices i, j).[18]

Contours of the true proof function derived from the OIML data set, together with contours of the slide rule compatible fit are presented in Fig. 2. (Imperial proof units were used because most of the hydrometer correcting slide rules produced use these units. Dicas's does not.) The dotted black curves are contours of the "exact" OIML function. Note that these generally have positive slope, but in the cold-wet region the slope is sometimes negative. In this region, violations of Eq. (3) are distinct. We may conclude that the "exact" function is not compatible with implementation on a slide rule without further approximation.

The solid blue curves in Fig. 2 are a best fit of the exact model using slide rule compatible Eq. (5). Contours of this fit match the exact function closely at high proof and high temperature, but the fit is quite poor in the cold-wet region where the exact function does not obey the necessary conditions.

The fitting calculation that produced Fig. 2 generated the functions u and v, and the mapping functions f, g, and h. Because the fiducial curve is horizontal, $h = f$ to within an additive constant. These mapping functions are displayed in Fig. 3 together with points digitized from the Dicas slide rule in Fig. 1.

Dicas measured proof using the system of Clarke's hydrometer in which 100 gallons of a spirit that is "x% above proof" becomes proof after addition of x gallons of water. Sikes's and modern systems use a different basis: 100 gallons of a "x% over proof" spirit can be diluted to give $100 + x$ gallons at proof. These systems are different because of the excess volume of mixing. Conversion of the digitized Dicas scale to an Imperial scale took this into account, together with Blagden's[16] measurement of Dicas proof spirit as having specific gravity 0.922 at 55°F.

The length of the scales was adjusted to make the distance between 30° and 80°F match. The exact correspondence of Dicas's temperature scale with the best fit in Fig. 3 is indicative of both scales being linear. In the model linearity is driven by the data, not model assumptions. Whether Dicas knew this or simply followed Occam's Razor is a mystery.

The frame of Dicas's slide rule has two proof scales that are unlabeled. The innermost scale corresponds best with a labeled "Dicas" scale on a slide rule produced by his son-in-law Benjamin Gammage.

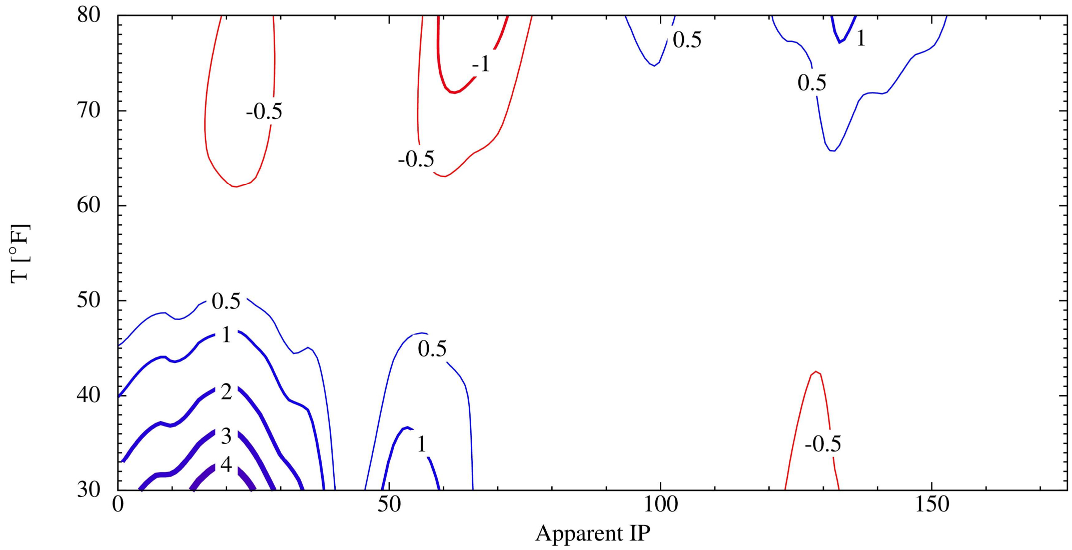

FIGURE 4. Contours of error: the true proof function deduced from the interpretation of Dicas' slide rule (Fig. 1) and an `exact´ calculation using the OIML data set.

Saunders,[19] who also produced a Dicas slide rule, put Dicas's scale inside and Clarke's outside. With these observations I will interpret the inner scale as corresponding to Dicas's system. Both scales are compressed by about 15% from the best fit scale. This suggests that Dicas's slide rule may have underestimated the necessary temperature correction.

The discrepancy between the inferred behavior of Dicas's calculation and the modern OIML one shown in Figure 4. It is quite small everywhere other than the cold-wet region. A hydrometer would rarely be relied on in this low proof region, since spirits have much higher proof and wines have dissolved sugars that render hydrometers unreliable. Dicas's slide rule is therefore quite accurate everywhere that matters.

As a basis of comparison, it is interesting to contrast the true proof correction using international standards (OIML and the thermal expansivity of glass) to the US Government's gauging tables. At 77°P (US) and 1°F (cold but not in the cold-wet region) the OIML calculation gives 100.74°P whereas the Alcohol Tobacco Tax and Trade Bureau (TTB) Gauging Table #1 gives 102.8°P. This difference, which rounds to 1.8°IP, is considerably larger than Dicas's error at the same point.

Conclusions

The conditions necessary for any function of two variables, $z(x, y)$, to be implemented on a slide rule have been derived. It is shown that, to within a multiplicative and an additive constant, the functions (f, g, h) that map variables (x, y, z) to distance may be deduced from the function $z(x, y)$, even when that function cannot be expressed in a closed form.

This analysis was applied to the problem of a hydrometer's temperature correction: the first known implementation of an empirical function on a slide rule, by John Dicas in 1780. While Dicas's data and his methods are unknown, his results are shown to be exceptionally accurate. Indeed, at points his slide rule correction is more accurate than the TTB gauging tables if more modern international standards are taken to be exact.

Acknowledgments

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Notes

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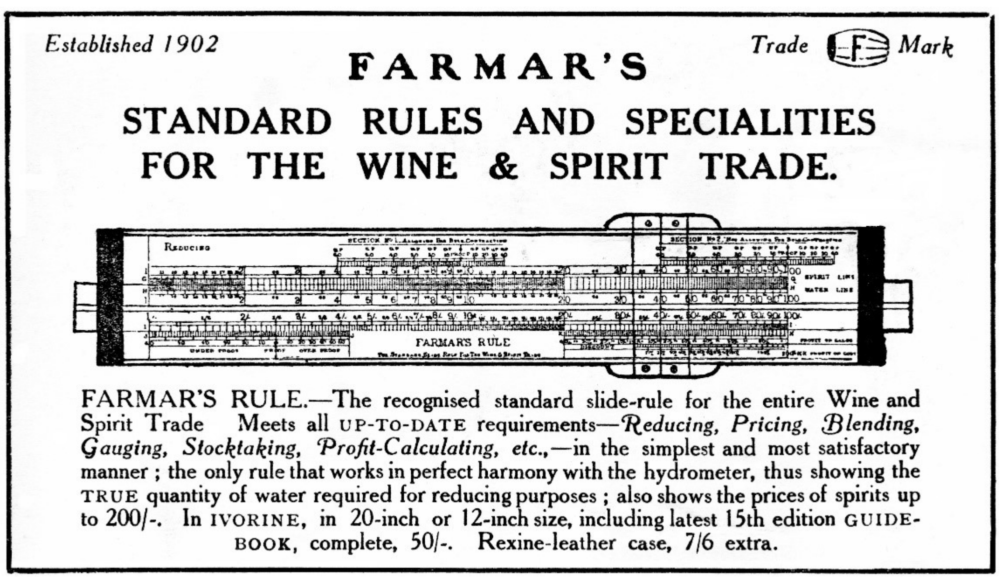

From "Guide to Farmar's Rule", 15th Edition, 1926